# Operations Research I

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Linear Programming and Duality

# Linear Programming

# Plan

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# Linear Programming

## Framework

**Linear Programming**

finite number of real variables, linear constraints, linear objective

Real variables $x_1, x_2 \dots x_n$

Constraint $(i)$:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

Objective function (maximize / minimize):

$$f(x_1, x_2 \dots x_n) = \sum_{j=1}^{n} c_j x_j$$

# Linear Programming

## Example: Cucumber and onions culture

Constraints about the quantities of fertilizer and anti-parasite

- $8\ell$ of fertilizer A available
  $\rightarrow 2\ell/m^2$ for cucumbers, $1\ell/m^2$ for onions
- $7\ell$ of fertilizer B available
  $\rightarrow 1\ell/m^2$ for cucumbers, $2\ell/m^2$ for onions
- $3\ell$ of anti-parasites available
  $\rightarrow 1\ell/m^2$ for onions

Objective: produce the maximum weight of vegetables knowing that the yield is $4kg/m^2$ for cucumbers, $5kg/m^2$ for onions

# Linear Programming

## Example: Cucumber and onions culture

**Decision variables**

- $x_c$: area of cucumbers
- $x_o$: area of onions

**Objective function** $\max 4x_c + 5x_o$

**Constraints**

- $2x_c + x_o \leq 8$ (fertilizer A)
- $x_c + 2x_o \leq 7$ (fertilizer B)
- $x_o \leq 3$ (anti-parasites)
- $x_c \geq 0$ and $x_o \geq 0$

# Linear Programming

**Interest**

General optimization problem with constraints

$\Rightarrow$ **NO GENERAL solution method!!**

Any linear problem

$\Rightarrow$ general and efficient solution methods

Those methods are efficient in theory and in practice

$\Rightarrow$ existence of numerous solution softwares:

Excel, CPLEX, Mathematica, LP-Solve...

**Restrictive framework**

- real variables
- linear constraints
- linear objective

# Linear Programming

**In extenso representation**

- $\max 4x_c + 5x_o$
- $2x_c + x_o \leq 8$ (fertilizer A)
- $x_c + 2x_o \leq 7$ (fertilizer B)
- $x_o \leq 3$ (anti-parasites)
- $x_c \geq 0$ and $x_o \geq 0$

**matrix representation**

$$\max \quad \begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} x_c \\ x_o \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ x_o \end{pmatrix} \leq \begin{pmatrix} 8 \\ 7 \\ 3 \end{pmatrix}$$

$$x_c \geq 0 \qquad x_o \geq 0$$

# Linear Programming

**in extenso representation**

$$\max z = \quad \sum_j c_j x_j$$

$$s.t. \qquad \sum_j a_{ij} x_j \quad \left\{ \begin{array}{c} \leq \\ \geq \\ = \end{array} \right\} \quad b_i \qquad i = 1, 2 \ldots m$$

$$x_j \qquad \geq \qquad 0 \qquad j = 1, 2 \ldots n$$

# Linear Programming

- second member $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$
- n decision variables $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$
- matrix $m \times n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- cost (or profit) $c = (c_1, c_2 \dots c_n)$

**Matrix representation**

$$\begin{aligned} \max z \quad &= cx \\ s.t. \qquad Ax \; &\left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} \; b \\ x \; &\geq \; 0 \end{aligned}$$

# Linear Programming

## Vocabulary

- $x_i$ **decision variable** of the problem
- $x = (x_1, \ldots, x_n)$ **feasible solution**
  *iff* it satisfies all constraints
- set of all feasible solutions = **admissible region**
- $x = (x_1, \ldots, x_n)$ **optimal solution**
  *iff* it is feasible and it optimizes the objective function

- **constraints** linear equalities or inequalities
  - $a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n \leq b_1$
  - $a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n \geq b_2$
  - $a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n = b_3$

- linear **objective function** (or economical function)
  - max / min $c_1x_1 + c_2x_2 \ldots + c_nx_n$

# Programmation linéaire

## Applications

**Exercice sheet:** Linear programming

- Wine production
- Advertizing
- Olive oil production
- Bergamote

**Caseine:** Lab Linear Programming

- Cucumbers and onions
- Perfumes
- Dairy Products
- Apples

# Linear Programming

## Canonical form of an LP

- maximization
- all variables are non negative
- all constraints are inequalities of the type "$\leq$"

$$\max z = \sum_j c_j x_j$$

$$s.t. \quad \sum_j a_{ij} x_j \leq b_i \qquad i = 1, 2 \dots m$$

$$x_j \geq 0 \qquad j = 1, 2 \dots n$$

- matrix form

$$\begin{aligned} \max z &= cx \\ s.t. \quad Ax &\leq b \\ x &\geq 0 \end{aligned}$$

# Linear Programming

**Standard form of an LP**

- maximization
- All variables are non negative
- All constraints are equations

$$\max z = \sum_j c_j x_j$$

$$\begin{array}{lrcll} s.t. & \sum_j a_{ij} x_j & = & b_i & i = 1, 2 \ldots m \\ & x_j & \geq & 0 & j = 1, 2 \ldots n \end{array}$$

- matrix form

$$\begin{array}{lrcl} \max z = cx & & & \\ s.t. & Ax & = & b \\ & x & \geq & 0 \end{array}$$

# Linear Programming

## Changing form

- equality $\rightarrow$ inequality

$$ax = b \iff \begin{cases} ax \leq b \\ ax \geq b \end{cases}$$

- max $\leftrightarrow$ min $\qquad \max f(x) = -\min -f(x)$
- inequality $\rightarrow$ equality: add a slack variable

$$\begin{array}{lll} ax \leq b & \iff & ax + s = b, \qquad s \geq 0 \\ ax \geq b & \iff & ax - s = b, \qquad s \geq 0 \end{array}$$

- unconstrained variable $\rightarrow$ positive variable

$$x \lesseqgtr 0 \iff \begin{cases} x = x^+ - x^- \\ \quad x^+, x^- \geq 0 \end{cases}$$

# Linear Programming

## Changing form

**Exercices sheet:** Linear Programming

- Linear and canonical forms

# Linear Programming

**Linearize a non linear formulation**

$e_i$: linear expression of the decision variables

- **obj:** $\min \max\{e_1, e_2 \ldots e_n\}$

$$\begin{cases} \min y \\ y \geq e_i \qquad i = 1, 2 \ldots n \end{cases}$$

- **obj:** $\max \min\{e_1, e_2 \ldots e_n\}$

$$\begin{cases} \max y \\ y \leq e_i \qquad i = 1, 2 \ldots n \end{cases}$$

- **obj:** $\min |e_1|$

$$|e_1| = \max(e_1, -e_1) \quad \begin{cases} \min y \\ y \geq e_1 \\ y \geq -e_1 \end{cases} \qquad \begin{cases} \min e^+ + e^- \\ e_1 = e^+ - e^- \\ e^+, e^- \geq 0 \end{cases}$$

# Linear Programming

## Linearize a non linear formulation

**Exercices sheet:** Linear Programming

- Linearization

# Linear Programming

## A little history

- 30's-40's: Kantorovitch, soviet economist
  $\Rightarrow$ linear models for production planning and optimization
- 40's-50's: Dantzig, american mathematician
  $\Rightarrow$ simplex algorithm
- historical application
  - Operation Vittles and Plainfare for supply of the trizone during the blockade of Berlin by airlift (June 23, 1948 has – May 12, 1949)
  - simplex executed by hand (thousands of variables), up to 12 000 tons of hardware each day!
- 1975: Kantorovitch has the Nobel price in economy
- XXIème century: software with LP available everywhere, use of LP in all industrial domains...

# Plan

# Geometrical interpretation

## Example: Cucumber and onions culture

**Decision variables**

- $x_c$: area of cucumbers
- $x_o$: area of onions

**Objective function** $\max 4x_c + 5x_n$

**Constraints**

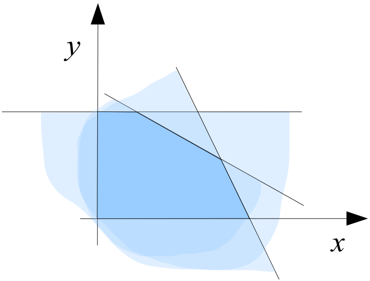
- $2x_c + x_n \leq 8$ (fertilizer A)
- $x_c + 2x_n \leq 7$ (fertilizer B)
- $x_n \leq 3$ (anti-parasites)
- $x_c \geq 0$ and $x_n \geq 0$

# Geometrical interpretation

**Interpret the constraints** cucumbers and onions

- $2x + y \leq 8 \Rightarrow$ half-plane of $\mathbb{R}^2$
- $x + 2y \leq 7 \Rightarrow$ half-plane
- $y \leq 3 \Rightarrow$ half-plane
- $x \geq 0$ and $y \geq 0 \Rightarrow$ half-plane

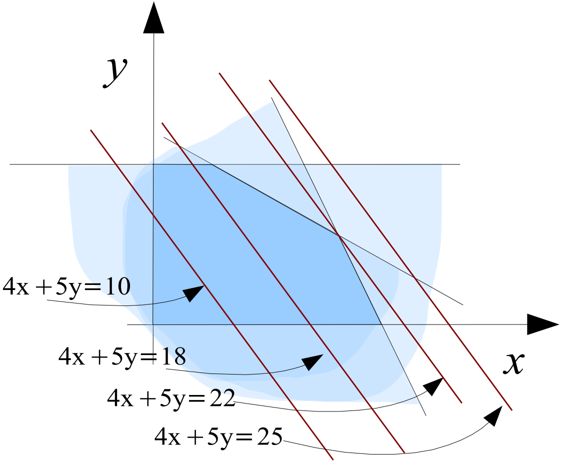
Set of feasible solutions = intersection of half-planes: **polyedron**



# Geometrical interpretation

**Optimize the objective**

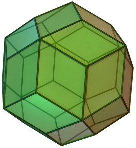
The **level lines** $\{4x + 5y = \text{constant}\}$ are parallel lines



# Geometrical interpretation

## Geometry of a PL

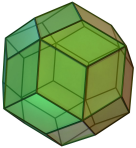
The set of feasible solutions is always a **polyhedron** (intersection of half-spaces)



The level lines $\{f = \text{constant}\}$ of the objective function $f$ are **affine hyperplanes** ($n = 2 \Rightarrow$ line, $n = 3 \Rightarrow$ plan...)

# Geometrical interpretation

**Geometry of a PL**



### Optimum is reached on the edge

The optimum of the objective function, if it exists, is reached on (at least) one of the **vertices** of the polyhedron

Mathematical justification:
the partial derivatives of $f(x) = c.x$ are never zero,
and the domain $\{x \mid \sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1, \ldots, m\}$ is compact
$\Rightarrow$ the optimum is reached on the edge...

# Linear Programming

**Solutions of an LP**

The feasible region can be

- empty
  - nb of optimal solutions: 0
- non empty, bounded
  - nb of optimal solutions: 1 or $\infty$
- non empty, unbounded
  - nb of optimal solutions: 0 or 1 or $\infty$

Propose examples of LP for each case

**Exercice sheet:** Linear programming

- Graphical solution

# Plan

# Basis and extreme points

**Recall**

$$\begin{array}{lrcl} \max & z & = & cx \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

- $A$ matrix $m \times n$
- $x = (x_1\, x_2 \ldots x_n)$
- $b = (b_1\, b_2 \ldots b_m)$
- $c = (c_1\, c_2 \ldots c_n)$

- The constraints define a polyhedron
- An optimal solution is a vertex of the polyhedron

How to enumerate the vertices of a polyhedron?

# Basis and extreme points

## Change to the standard form

### Standard form

Add **slack variables**:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \;\Leftrightarrow\; \sum_{j=1}^{n} a_{ij}x_j + e_i = b_i, e_i \geq 0$$

Standard LP:

$$\begin{array}{lccc} \max & z(x) & = & c.x \\ \text{s.c} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

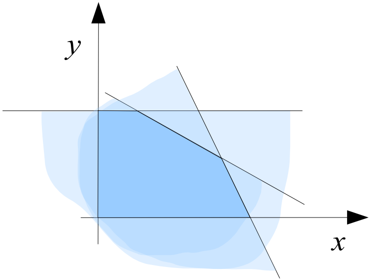
We work on a space of higher dimension but all constraints are equalities

- Easier algebraic manipulations

# Basis and extreme points

## Change to the standard form

$$\begin{aligned} \max z &= 4x + 5y \\ \text{s.t.} \quad & 2x + y \leq 8 \\ & x + 2y \leq 7 \\ & y \leq 3 \\ & x, y \geq 0 \end{aligned}$$




$$\begin{aligned} \max z &= 4x + 5y \\ \text{s.t.} \quad & 2x + y + e_1 = 8 \\ & x + 2y + e_2 = 7 \\ & y + e_3 = 3 \\ & x, y, e_1, e_2, e_3 \geq 0 \end{aligned}$$

9 interesting points
(intersection of constraints)

5 feasible points

enumeration of those 9 points as solutions of the standard form (basic solution)

# Basis and extreme points

$$
\begin{array}{lrcrcrcrcrcl}
\text{s.t.} & 2x & + & y & + & e_1 & & & & & = & 8 \\
 & x & + & 2y & & & + & e_2 & & & = & 7 \\
 & & & y & & & & & + & e_3 & = & 3 \\
 & x, & & y, & & e_1, & & e_2, & & e_3 & \geq & 0
\end{array}
$$

| x | y | $e_1$ | $e_2$ | $e_3$ | basic solution | admiss. | extreme pt |
|---|---|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{0}$ | 8 | 7 | 3 | ✔ | ✔ | (0,0) |
| $\underline{0}$ | 8 | $\underline{0}$ | -9 | -5 | ✔ | ✘ | |
| $\underline{0}$ | 3.5 | 4.5 | $\underline{0}$ | -0.5 | ✔ | ✘ | |
| $\underline{0}$ | 3 | 5 | 1 | $\underline{0}$ | ✔ | ✔ | (0,3) |
| 4 | $\underline{0}$ | $\underline{0}$ | 3 | 3 | ✔ | ✔ | (4,0) |
| 7 | $\underline{0}$ | -6 | $\underline{0}$ | 3 | ✔ | ✘ | |
| | $\underline{0}$ | | | $\underline{0}$ | ✘ | ✘ | |
| 3 | 2 | $\underline{0}$ | $\underline{0}$ | 1 | ✔ | ✔ | (3,2) |
| 2.5 | 3 | $\underline{0}$ | -1.5 | $\underline{0}$ | ✔ | ✘ | |
| 1 | 3 | 3 | $\underline{0}$ | $\underline{0}$ | ✔ | ✔ | (1,3) |

$\{$**extreme points**$\} \iff \{$**feasible basic solutions**$\}$

# Basis and extreme points

- Linear system $Ax = b$
- $A$ format $m \times n$, rank $A = m \leq n$
- **Basis** of $A$: invertible submatrix $B(m \times m)$ of $A$
  $A = (B, N)$

$$(B, N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \qquad \text{or} \qquad Bx_B + Nx_N = b$$

$$\Rightarrow \quad x_B = B^{-1}b - B^{-1}Nx_N$$

- **Basic solution** associated to $B$:
  - $x_N = 0$ non-basic variables
  - $x_B = B^{-1}b$ basic variables

# Basis and extreme points

## Applications

**Exercice sheet:** Linear programming

- Exercice Bases *2

# Basis and extreme points

**Basis and basic solutions**

$$\begin{cases} 2x + y + e_1 = 8 \\ x + 2y + e_2 = 7 \\ y + e_3 = 3 \\ x, y, e_1, e_2, e_3 \geq 0 \end{cases}$$

Initial basis ? $\{e_1, e_2, e_3\}$ for example:

$$\begin{cases} 2x + y + e_1 = 8 \\ x + 2y + e_2 = 7 \\ y + e_3 = 3 \end{cases} \Leftrightarrow \begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases}$$

$e_1, e_2, e_3 =$ basic variables, $x, y =$ non-basic variables

# Basis and extreme points

## Basis and basic solutions

$$\begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases}$$

- non-basic variables are set to 0
- basic variables are then calculated

$$x = y = 0 \Rightarrow \begin{cases} e_1 = 8 - 2x - y = 8 \\ e_2 = 7 - x - 2y = 7 \\ e_3 = 3 - y = 3 \end{cases}$$

# Basis and extreme points

- $Ax = b, \qquad x \geq 0$
- $(x_B, 0)$ associated to $B$ is a **feasible basic solution** if $x_B \geq 0$
- $\{$**extreme points** of the polyhedron$\}$ $\Longleftrightarrow$ $\{$feasible basic solutions of the corresponding linear system$\}$
- number of extreme points $\approx C_n^m = \frac{n!}{m!(n-m)!}$
- basic degenerated solutions: some basic variables are zero
- if $A$ is invertible: a single basic solution

# Basis and extreme points

## Neighboring basis and pivoting

### Neighboring basis

Two neighboring vertices correspond to two bases $B$ and $B'$ such that a variable of $B$ is replaced to obtain $B'$

- pass to a neighboring vertex: change basis (neighboring basis)

**pivoting principle**

# Basis and extreme points

**Which variable enters the basis?**

Let's try with $y$: what is the max value for $y$?

- $e_1 = 8 - 2x - y \geq 0 \Rightarrow y \leq 8$
- $e_2 = 7 - x - 2y \geq 0 \Rightarrow y \leq 3.5$
- $e_3 = 3 - y \geq 0 \Rightarrow y \leq 3$

$y_{\text{max}} = 3$, for $y = y_{\text{max}}$ one has $e_1 = 5 - 2x$, $e_2 = 1 - x$, and $e_3 = 0$

▶ candidate for a new basis: $\{e_1, e_2, e_3\} \cup \{y\} \setminus \{e_3\} = \{e_1, e_2, y\}$

$(x, y, e_1, e_2, e_3) = (0, 3, 5, 1, 0)$

# Plan

# The simplex algorithm

**Towards a solution algorithm**

▶ A naive solution method: enumerate all vertices, calculate $f$ on these points, choose the vertex with optimal $f$:

- works: finite number of vertices
- limitation: this number can be very large in general...

**The simplex algorithm** (G. B. Dantzig 1947) iterative algorithm allowing to solve a linear program.

# The simplex algorithm

## Local improvement principle

From a current vertex, find a neighboring vertex that improves the objective.

## Local improvement principle (maximization):

Let $x_0$ be a non optimal vertex. Then, there exists $x$, a **neighboring** vertex of $x_0$, such that $f(x) > f(x_0)$.

- Solution methods: start from any vertex $x_0$, move to a neighboring vertex for which $f$ increases and so on.

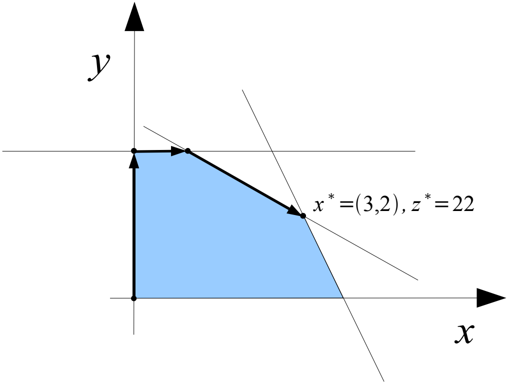
Remark: we change from a **continuous** problem (real variables) to a **discrete** problem (finite number of vertices)...

# The simplex algorithm

**2D Illustration: Cucumber and onions**

$z = 4x + 5y$

$x_0 = (0,0), z = 0 \rightarrow x = (0,3), z = 15$
$x_0 = (0,3), z = 15 \rightarrow x = (1,3), z = 19$
$x_0 = (1,3), z = 19 \rightarrow x = (3,2), z = 22$




- no more local improvement is possible ⇒ optimum

# The simplex algorithm

**Concrete illustration**

- Standardization:

$$\text{Maximize } z = 4x + 5y \quad \text{s.t.} \begin{cases} 2x + y \leq 8 \\ x + 2y \leq 7 \\ y \leq 3 \\ x, y \geq 0 \end{cases}$$

$$\text{Maximize } z = 4x + 5y \quad \text{s.t.} \begin{cases} 2x + y + e_1 = 8 \\ x + 2y + e_2 = 7 \\ y + e_3 = 3 \\ x, y, e_1, e_2, e_3 \geq 0 \end{cases}$$

Initial basis? $\{e_1, e_2, e_3\}$ for example:

$$\begin{cases} 2x + y + e_1 = 8 \\ x + 2y + e_2 = 7 \\ y + e_3 = 3 \end{cases} \Leftrightarrow \begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases}$$

$e_1, e_2, e_3 =$ basic variables, $x, y =$ non-basic variables

# The simplex algorithm

**Associated basic solution**

- set the non-basic variables to 0
- We deduce:
  - value of the basic variables
  - value of $z$

$$\text{here: } x = y = 0 \Rightarrow \begin{cases} e_1 = 8 - 2x - y = 8 \\ e_2 = 7 - x - 2y = 7 \\ e_3 = 3 - y = 3 \end{cases} \text{ and } z = 4x + 5y = 0$$

# The simplex algorithm

**Basis change**

Essential observation: $z = 4x + 5y = 0 \Rightarrow$ we can improve $z$ if $x$ or $y$ enters in the basis.

Let's try with $y$: What is the maximum value for $y$ ?

- $e_1 = 8 - 2x - y \geq 0 \Rightarrow y \leq 8$
- $e_2 = 7 - x - 2y \geq 0 \Rightarrow y \leq 3.5$
- $e_3 = 3 - y \geq 0 \Rightarrow y \leq 3$

$y_{\max} = 3$, for $y = y_{\max}$ we have $e_1 = 5 - x$, $e_2 = 1 - x$, and $e_3 = 0$

▶ candidate for a new basis: $\{e_1, e_2, e_3\} \cup \{y\} \setminus \{e_3\} = \{e_1, e_2, y\}$

# The simplex algorithm

**New basis** $\{e_1, e_2, y\}$

$$\begin{cases} e_1 = 8 - 2x - y \\ e_2 = 7 - x - 2y \\ e_3 = 3 - y \end{cases} \Rightarrow \begin{cases} e_1 = 8 - 2x - y = 5 - 2x + e_3 \\ e_2 = 7 - x - 2y = 1 - x + 2e_3 \\ y = 3 - e_3 \end{cases}$$

Describe $z$ with the non-basic variables

- $z = 4x + 5y = 15 + 4x - 5e_3$

Associated basic solution

$$x = e_3 = 0 \Rightarrow \begin{cases} e_1 = 5 - 2x + e_3 = 5 \\ e_2 = 1 - x + 2e_3 = 1 \\ y = 3 - e_3 = 3 \end{cases} \quad \text{and} \quad z = 15$$

# The simplex algorithm

**Iteration**

$z = 15 + 4x - 5e_3$ can still increase if $x$ enters the basis

If $x$ enters, which variable leaves the basis?
Max value for $x$:

- $e_1 = 5 - 2x + e_3 \geq 0 \Rightarrow x \leq 2.5$
- $e_2 = 1 - x + 2e_3 \geq 0 \Rightarrow x \leq 1$
- $y = 3 - e_3 \geq 0 \Rightarrow$ no constraint on $x$

Finally: $x_{\max} = 1$ and $e_2$ leaves the basis
New basis $\{e_1, y, x\}$

$$\begin{cases} e_1 = 3 + 2e_2 - 3e_3 \\ x = 1 - e_2 + 2e_3 \\ y = 3 - e_3 \\ z = 19 - 4e_2 + 3e_3 \end{cases}$$

# The simplex algorithm

**Iteration (cont.)**

$z = 19 - 4e_2 + 3e_3$ can still increase if $e_3$ enters the basis

If $e_3$ enters, which variable leaves the basis?
Max value for $e_3$:

- $e_1 = 3 + 2e_2 - 3e_3 \geq 0 \Rightarrow e_3 \leq 1$
- $x = 1 - e_2 + 2e_3 \geq 0 \Rightarrow$ no constraint on $e_3$
- $y = 3 - e_3 \geq 0 \Rightarrow e_3 \leq 3$

Finally: $e_{3_{\max}} = 1$, $e_1$ leaves. New basis $\{e_3, y, x\}$:

$$\begin{cases} e_3 = 1 + 2/3e_2 - 1/3e_1 \\ x = 3 + 1/3e_2 - 2/3e_1 \\ y = 2 - 2/3e_2 + 1/3e_1 \\ z = 22 - 2e_2 - e_1 \end{cases}$$

# The simplex algorithm

**Termination**

One has $z = 22 - 2e_2 - e_1$, hence $z^* \leq 22$

but the basic solution $x = 3, y = 2, e_3 = 1$ leads to $z = 22$

- optimum

The termination conditions concern the coefficients of $z$ expressed with the non-basic variables

# The simplex algorithm

**Reduced costs**

$B$, a basis of $Ax = b$

the objective function

$$\begin{aligned} z &= cx = c_B x_B + c_N x_N \\ &= c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N \\ &= z_0 - \sum_{j=1}^{n} (c_B B^{-1} a^j - c_j) x_j \\ &= z_0 - \sum_{j=1}^{n} (z_j - c_j) x_j \end{aligned}$$

$z_j - c_j = c_B B^{-1} a^j - c_j$ are the reduced costs of the non-basic variable $x_j$

# The simplex algorithm

At each iteration

| | | | $x_N$ | $x_B$ |
|---|---|---|---|---|
| $z$ | $=$ | $z_0$ | reduced costs | $0$ |
| $x_B$ | $=$ | $\oplus$ | $\ddots$ | |

For the optimum

| | | | $x_N$ | $x_B$ |
|---|---|---|---|---|
| $z$ | $=$ | $z_0^*$ | $\ominus$ | $0$ |
| $x_B$ | $=$ | $\oplus$ | $\ddots$ | |

# The simplex algorithm

Heuristic: the variable with the higher coefficient enters the basis

Which variable leaves the basis?

**Minimal quotient principle**

| pivot column $x_1$ | right member $\geq 0$ | quotient |
|---|---|---|
| $a_1 > 0$ | $b_1$ | - |
| $a_2 < 0$ | $b_2$ | $-\frac{b_2}{a_2}$ |
| $a_3 < 0$ | $b_3$ | $-\frac{b_3}{a_3}$ |
| $a_4 = 0$ | $b_4$ | - |

row $r$ $\quad \frac{b_r}{a_r} = \min \left\{ -\frac{b_i}{a_i} | a_i < 0 \right\}$

# The simplex algorithm

**Pivoting**

$$
\begin{array}{rccc}
 & c & \cdots & a \\
 & \vdots & & \vdots \\
\text{pivot row} - & p & \cdots & b \\
 & | & & \\
 & \text{pivot} & & \\
 & \text{column} & &
\end{array}
\implies a \to a - \frac{b}{p} c
$$

# The simplex algorithm

**Phase II**

Input: a linear program and a feasible basic solution
Output: an optimal feasible solution or declare "non-bounded linear program"

1. Choose an entering column (variable)
    - choose a non-basic variable $x_j$ (column) with a negative reduced cost
    - if there is no entering variable: STOP, the basic solution is optimal
2. Choose a leaving row (variable)
    - choose a row $r$ minimizing the quotient
    - if there is no leaving row: STOP the current linear program is unbounded
3. Update the basis and the program
    - pivot around $a_{rj}$ and go to (1)

# The simplex algorithm

- A basic solution is degenerated if at least one basic variable is zero (in this case, there isn't a bijection between the feasible basic solutions and the extreme points)
- If all basic solutions are non degenerated, then the simplex algorithm terminates after a finite number of iterations

# The simplex algorithm

**Phase I**

Phase II of simplex algorithm takes as an input a feasible basis.
Phase I allows to find a first feasible basis.

**Discover Phase I**

- **Caseine:** Phase I*
- **Exercice sheet:** Simplexe *2

# Duality

# Plan

5. Economical Illustration
6. How to prove optimality?
7. Write the dual
8. Properties

# Duality

**New concept in Linear Programming**

**Primal**

- input $A$, $b$, $c$
- minimize

**Dual**

- same input $A$, $b$, $c$
- maximize

# Plan

# Plan

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# Primal problem ($\mathcal{P}$)

A family uses 6 food products
as a source of vitamin A and C

| | products (units/kg) | | | | | | demand (units) |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | |
| vitamin A | 1 | 0 | 2 | 2 | 1 | 2 | 9 |
| vitamin C | 0 | 1 | 3 | 1 | 3 | 2 | 19 |
| Price per kg | 35 | 30 | 60 | 50 | 27 | 22 | |

**Objective :** minimize the total cost

Modelization

# Dual problem $(\mathcal{D})$ associated to $(\mathcal{P})$

A producer of tablets of synthetic vitamin wants to convince the family to by his vitamins.

**What are the selling prices $w_A$ and $w_C$ ?**

- to be competitive
- and maximize the profit

Modelization

# matrix modelling

**Primal problem**

the family: buy food products at the minimum price to satisfy the need in vitamin A and C.

Matrix modelling

**Dual problem**

Synthetic vitamin producer: be competitive with the food products as a source of vitamin and maximize the sale profit

Matrix modelling

# Generalization of the economic illustration

| | resource $i$ | demand $j$ |
|---|---|---|
| product $j$ | $a_{ij}$ | $c_j$ |
| cost $i$ | $b_i$ | |

**Primal problem** (product purchaser): What quantity of resource $i$ has to be bought to satisfy the demand at minimum cost?

$$\min \sum_i b_i x_i \qquad s.t. \quad \sum_i a_{ij} x_i \geq c_j \quad \forall j$$

**Dual problem** (seller of the product): What price should product $j$ be proposed at to maximize the profit while remaining competitive?

$$\max \sum_j c_j w_j \qquad s.t. \quad \sum_j a_{ij} w_j \leq b_i \quad \forall i$$

# Plan

# How to prove optimality?

Objective: prove the optimality of a solution

$$\max z = x_1 + x_2$$
$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$
$$x_1, x_2 \geq 0$$

**Idea:** find a valid combination of the constraints that bounds each term of the objective function

# How to prove optimality?

$\max z = x_1 + x_2$

$$
\begin{array}{rcrcll}
4x_1 & + & 5x_2 & \leq & 20 & \times y_1 \\
2x_1 & + & x_2 & \leq & 6 & \times y_2 \\
 & & x_2 & \leq & 2 & \times y_3 \\
\hline
(4y_1 + 2y_2)x_1 & + & (5y_1 + y_2 + y_3)x_2 & \leq & 20y_1 + 6y_2 + 2y_3 & \\
 & & & \uparrow & & \\
 & & & y_1, y_2, y_3 \geq 0 & &
\end{array}
$$

Finally,

$$
\begin{array}{lrcrcrcl}
\min & 20y_1 & + & 6y_2 & + & 2y_3 & & \\
\text{s.t.} & & & & & & & \\
 & 4y_1 & + & 2y_2 & & & \geq & 1 \\
 & 5y_1 & + & y_2 & + & y_3 & \geq & 1 \\
 & y_i & \geq & 0 & & & &
\end{array}
$$

(minimal upper bound)

(bound each term of the objective fu

# Plan

5 Economical Illustration

6 How to prove optimality?

7 Write the dual

8 Properties

# Canonical form of Duality

Input $A$, $b$, $c$

$$(\mathcal{P}) \quad \begin{cases} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{cases}$$

$$(\mathcal{D}) \quad \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{cases}$$

# Sign table

| min | max |
|---|---|
| primal | dual |
| dual | primal |
| variable $\geq 0$ | constraint $\leq$ |
| variable $\lesseqgtr 0$ | constraint $=$ |
| variable $\leq 0$ | constraint $\geq$ |
| constraint $\leq$ | variable $\leq 0$ |
| constraint $=$ | variable $\lesseqgtr 0$ |
| constraint $\geq$ | variable $\geq 0$ |

Writing the dual is automatic:

- the variables
- the objective function
- the constraints

# Write the dual

Write the dual program of

$\max z = 4x_1 + 5x_2 + 2x_3$

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & & & = & 3 \\ & & & & 2x_3 & \geq & 2 \\ 3x_1 & + & x_2 & + & x_3 & \leq & 2 \\ & & x_2 & + & x_3 & \leq & 1 \end{array}$$

$x_1 \geq 0 \quad x_2 \leq 0 \quad x_3 \geq 0$

# Plan

5. Economical Illustration

6. How to prove optimality?

7. Write the dual

8. Properties

# Properties

## Property

The dual of the dual is equivalent to the primal

verify on an example

$$\max z = 2x_1 + 3x_2 + 4x_3$$

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & & & \leq & 3 \\ & & & & x_3 & \geq & 2 \\ 3x_1 & + & x_2 & + & x_3 & \leq & 2 \\ & & x_2 & & & \leq & 1 \end{array}$$

$$x_1, x_2 \geq 0, \qquad x_3 \leq 0$$

# Properties

$$(\mathcal{P}) \begin{cases} \min & z = cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{cases} \qquad (\mathcal{D}) \begin{cases} \max & v = wb \\ s.t. & wA \leq c \\ & w \geq 0 \end{cases}$$

## Weak duality theorem

For each pair of feasible solutions $x$ of $(\mathcal{P})$ and $w$ of $(\mathcal{D})$

$z = cx \geq wb = v$

Consequence: what if one of them is not bounded?

# And optimality?

## Optimality certificate

If

$$z = cx = wb = v$$

for feasible solutions $x$ of $(\mathcal{P})$ and $w$ of $(\mathcal{D})$, then $x$ and $w$ are optimal.

## Strong duality theorem

If $(\mathcal{P})$ has solutions and $(\mathcal{D})$ has solutions, then

$$cx^* = w^* b$$

# Complementary slackness property

For the vitamins example

- write the primal with the slack variables $(s_i)$
- write the dual with the slack variables $(t_i)$
- find a primal optimal solution
- find a dual optimal solution
- write the pairs of variables $(s_i, w_i)$ and $(x_j, t_j)$
- can you notice something?

# Complementary slackness property

## Complementary slackness property

$x^*$ optimal for $(\mathcal{P})$ and $w^*$ optimal for $(\mathcal{D})$ verify

- the slack variable of a constraint of $(\mathcal{P})$ is zero
  OR
- the variable associated with this constraint in $w^*$ is zero

likewise in the other way round

$$x_j t_j = 0 \text{ and } s_i w_i = 0$$

Proof

# Complementary slackness property

**Interest** Knowing an optimal solution $x^*$ of $(\mathcal{P})$, then $y^*$ can be found by applying the complementary slackness property (thus proving optimality of $x^*$)

try on an example

$\max z = x_1 + x_2$

$$\begin{array}{rcrcl} 4x_1 & + & 5x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 6 \\ & & x_2 & \leq & 2 \end{array}$$

$x_1, x_2 \geq 0$

with $x_1 = 2$ and $x_2 = 2$

# A small philosophy of Duality

What is the interest of the three theorems of duality

- Weak duality: to make the **proof of optimality**
- Complementary slackness: to find an optimal solution of the dual knowing an optimal solution of the primal
- Strong duality: **guaranties** that an optimality proof is feasible (using duality)